CORRESPONDENCE.

Editor Analyst:

OWING to absence from home, the answers published in the ANALYST in July 1881, to a query proposed by me escaped my notice until quite recently. I trust the subject will be found to possess sufficient interest for your readers to justify a recurrence to it. The query was as follows:—

"Let $u = \frac{\sin ax}{a}$. Now if $a = \infty$, $u = 0$ independently of x, therefore we should have $\frac{du}{dx} = 0$, when $a = \infty$. But $\frac{du}{dx} = \cos ax$, which is essentially indeterminate when $a = \infty$. What is the explanation of this paradox?"

Prof. Barbour, Mr. Adcock and Prof. Judson replied on page 129, ANALYST, Vol. VIII. Prof. Barbour does not seem to have considered the paradox as intended; for he does not make a infinite, but merely points out that $u = 0$ whenever ax is a multiple of 2π, while $\frac{du}{dx} = 1$, and remarks that u may $= 0$ while $\frac{du}{dx} = 1$. This is of course true, but the paradox consists in the fact that $u = 0$ for *all* values of x in the case considered; hence u being a constant so far as x is concerned, we should expect to find $\frac{du}{dx} = 0$ for all values of x.

Mr. Adcock remarks that "when $u = 0$ independently of x, it is not a function of x, and *therefore cannot be differentiated with respect to* x. Therefore the value of $du \div dx$ is $\cos ax$ independently of the value of a." I do not understand Mr. Adcock to mean by the words which I have italicized that $du \div dx$ ceases to have a meaning when u ceases to be a function of x. The natural conclusion would seem to be that the value of $du \div dx$ which is ordinarily a function of x, should become zero when u ceases to be a function of x. For example, $u = a^x$ ceases to be a function of x (at least for all finite values of x) when $a = 1$, viz., it takes the value 1 independently of x; accordingly we find that $\frac{du}{dx} = \log a \,.\, a^x$ becomes zero when $a = 1$. Assuming the a at the end of the last sentence to be misprinted for x, Mr. Adcock seems to draw only the conclusion that the value of $du \div dx$ should become independent of x; but he does not deny that the form assumed, namely $\cos \infty$, is essentially indeterminate, and not, as in the case instanced above, always equal to zero.

Prof. Judson, on the other hand, says "If $u=0$ independently of x, then u is not a function of x, and $du \div dx$ is without meaning." It is not necessary to discuss this point, for we already have an expression for the value of $du \div dx$, and the matter in hand is to account for the fact that this expression does not assume the value zero under the circumstances, as would naturally be expected. Prof. Judson, however, goes on to say "If a is a constant, then a cannot $= \infty$. If a is a variable independent of x, and $a = \infty$, i. e., a increases without limit, then $(\sin ax) \div a$ is an infinitesimal (not$=0$) *and* u *is therefore indeterminate*; $du \div dx$ is also indeterminate, and there is no paradox." It is of course sufficient to regard a as independent of x, and no objection will be made to Prof. Judson's phraseology. I have however italicized one clause for the sake of the remark that, if we admit that an infinitesimal is in a certain sense indeterminate, it is not necessarily or usually so in a sense that would imply that (as in the case of an indeterminate *finite* quantity) its derivative should admit of finite values. Wherein does this case differ from that of $u = a^x$, already mentioned? Not in the fact that u is infinitesimal for we might have taken $u = \dfrac{a + \sin ax}{a}$ whose value approaches indefinitely to 1 as a incereases without limit; nor in the fact that the critical case occurs when a (in the ordinary phraseology) $= \infty$; for we might have written $u = (a - 1) \sin \dfrac{x}{a-1}$, and the critical case would have occurred when $a = 1$.

If u be made the ordinate of a curve of which x is the abscissa, then for any finite value of a the curve is a sinusoid, and as a increases the waves become smaller both in length and amplitude. As a increases without limit, the curve approaches indefinitely to the axis of x, whose equation is $u = 0$, whence we should naturally expect that $du \div dx$ would be zero, but since the waves do not change their shape when they become infinitesimal, we find that $du \div dx$ still admits of the same values as when a is finite, viz., all values between $+1$ and -1.

The analytical difficulty is connected with the occurrence of the form $\cos \infty$. I have not hesitated in the statement of the paradox to regard this as an essentially indeterminate form. In an interesting memoir "On the Sine and Cosine of an Infinite Angle", *Cambridge Philosophical Transactions*, Vol. VIII, p. 255, Mr. S. Earnshaw contends for the indeterminateness of these forms in opposition to writers upon Definite Integrals, who, he says, while admitting "that when x becomes infinite $\sin x$ and $\cos x$ cannot be said to be in one part of their periodicity rather than another" yet agree in "practically affirming that both the sine and cosine of an infinite angle are

equal to zero." In this memoir Mr. Earnshaw points out the falacies in the proofs that have been given of the equation $\cos\infty = 0$. In the same vol., De Morgan had, at page 191, given reasons why we should expect periodic functions, when indeterminate, to be represented by their mean values, and remarks that the indeterminate symbols, $\sin\infty$ and $\cos\infty$, are found in numberless cases to represent, each of them, 0, the mean value of both $\sin x$ and $\cos x$. Mr. J. W. L. Glaisher also discusses this question in the 5th volume of the first series of the *Messenger of Mathematics* with a view of determining the conditions under which these expressions may be taken equal to zero; or more generally, under which a periodic function may be assumed equal (when x is infinite) to its mean value, or φ being a rational function,

$$\varphi(\sin\infty, \cos\infty) = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\sin x, \cos x)dx.$$

The paradox is not without interest inasmuch as the geometrical illustration renders it evident that the effective value of $du \div dx$ ought to coincide with its mean value.

In this connection I may also mention that the difficulty arising in the application of the theorem

$$\frac{f(x)}{\varphi(x)} = \frac{f'(x)}{f'(x)} \text{ to the example } \frac{x - \sin x}{x + \cos x},$$

when $f(x) = \infty$, $\varphi(x) = \infty$ when $x = \infty$, given in Bertrand's Calcul Differential, p. 476 (quoted in Rice and Johnson's Calculus, p. 114), disappears if we admit that $\sin\infty$ and $\cos\infty$ are each equal zero.

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GEOMETRICAL DETERMINATION OF THE SOLIDITY OF THE PARABOLA.

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LET AD be a parbola inscribed in the parallelogram $ALDR$. Suppose RCB to be one of an infinite number of inscribed triangles. Through C and B draw Mm and NK respectively parallel to RA, and draw CPH perpendicular to NK. Through E, the middle point of CB, draw the tangent ST to meet RS a perpendicular to it in S; also through E draw Oo parallel to Mm, and VUF perpendicular to RA.

From the similar triangles PCB and SRT we have

$$PC : CB :: RS : RT.$$

$$\therefore PC \times RT = RS \times CB.$$